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# Development of analytical vibration solutions for microstructured beam model to calibrate length scale coefficient in nonlocal Timoshenko beams

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The present study takes an analytical approach for solving the free vibration problem of a microstructured beam model, in which transverse displacement springs are added to allow for the transverse shear deformation effect in addition to the rotational springs. The exact vibration frequencies for the discrete microstructured beam model with simply supported ends are obtained via matrix decomposition. In addition, a general solution technique involving the use of Padé approximants for the continualization procedure is proposed in order to obtain the continuous equivalent system for the discrete microstructured beam model. The analytical vibration solutions of the equivalent continuous system are obtained and their accuracy is assessed by using the exact solutions. It is found that the solutions of the equivalent continuous system have a first order accuracy when compared with the exact solutions of their discrete counterpart. The length scale coefficient in the nonlocal Timoshenko beam model is calibrated by using the analytical solutions. Two nonlocal Timoshenko beam models, i.e., the Wang model (without the length scale effect in the shear stress strain relation) and the Reddy model, are evaluated based on their ability to capture the nonlocal effect. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4820565]

## I. INTRODUCTION

The repetitive cell (or unit cell) structures exist in multiscale: from the atomic scale, for instance, in describing a molecule of a polymer, a crystalline of a polycrystal or a grain of a granular material (Mindlin, 1964), to the macroscale, such as lattice structures spanning large areas with few intermediate supports and large-area space structures (Ostoja-Starzewski, 2002). The key characteristic of these repetitive cell structures is that a unit cell (or a pattern) fills in the space of the structure, which results in the "non-homogenous" and "non-continuum" feature from a materials point of view.

The approaches for solving these repetitive cell structures problems (Bažant and Christensen, 1972) include (a) applying finite difference calculus and obtaining the exact solutions/series solutions or (b) making a continuous approximation. The continuous approximation could be conducted either on the motion of equation (or term as "continualization") via the Taylor series (Mindlin, 1972), or more accurately by the Padé approximants (Andrianov and Awrejcewicz, 2001) or on the constitutive law for instance Eringen's nonlocal theory (Eringen and Edelen, 1972; Eringen, 1983), strain gradient theory (Mindlin 1964; Askes and Aifantis, 2011) and homogenization of a representative volume element (Odegard *et al.*, 2002).

Approach (a) has been explored for certain cases by many authors. For example, the stress, deflection and buckling analysis of two-dimensional lattices of flexural members were obtained by (Dean and Ugarte, 1968). A discrete system with infinite shear stiffness has been proposed to solve the free vibration problem (in axial and bending and direction, see the work by Elishakoff and Santoro (2005); Santoro and Elishakoff (2006); and Livesley (1955)) and forced bending vibration response by Gürgöze and Özer (1994). It should be noted that such discrete systems have been referred to as coarse lattice model or spring network (Odegard *et al.*, 2002). However in order to avoid confusing with the true lattice model or atomic lattice model where the model consists of nodal masses connected by springs with one another, "microstructured beam model" is proposed and adopted in the current study. Recently, Challamel *et al.* (2013) investigated the buckling of a microstructured column with simply supported ends via a Lagrange multiplier. It is found that the exact buckling load of the finite degree-of-freedom column may be expressed as a recursive formula involving Chebyshev polynomials. Prompted by the findings in Challamel *et al.* (2013), a matrix decomposition is proposed herein to solve the free vibration problem of a microstructured beam model and to link the exact solutions with the second kind Chebyshev polynomials.

For a continuous approximation on the motion of equation (or "continualization") in approach (b), the Taylor series (Filimonov, 1996) and the Padé approximants (Andrianov *et al.*, 2010) have been used to solve the wave equation for a one-dimensional 1D chain of particles (or axial spring) (Askar, 1986), transverse motions of a 1D chain of dumbbell particles (Noor and Nemeth, 1980), bending and vibration of

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beamlike and platelike lattices (Noor *et al*., 1978), beam bending (Andrianov and Awrejcewicz, 2005; Andrianov *et al*., 2010), and buckling of beamlike lattices (Challamel *et al*., 2013). Recently, Challamel *et al*. (2013) proposed a Padé approximant to convert the fourth order finite difference operator to a differential counterpart. This Padé approximant is extended to fulfil the continualization in the current study so that the discrete microstructured beam model is transformed into a continuous system and consequently analytical solutions are possible.

As one of the continuous approximations on the constitutive law in approach (b), both Eringen's nonlocal theory (Eringen and Edelen, 1972 and Eringen, 1983) and strain gradient theory (Askes and Aifantis, 2011) have been widely used for analysing repetitive cell structures in nano/micro scales, especially for carbon nanotubes (CNTs) and graphene sheets. Nonlocal structural models, including nonlocal shell, plate, and beam models, have developed rapidly and a detailed survey of this line of investigation is presented in the review papers by Askes and Aifantis (2011) and Arash and Wang (2012). The application of such a nonlocal theory to beam structures like CNTs is extensively investigated including bending, buckling, vibration, and wave propagation analysis (see, for instance, Peddieson *et al*. (2003); Sudak (2003); Wang and Hu (2005); Wang (2005); Wang *et al*. (2006); Wang *et al*. (2007); Wang and Liew (2007); Wang and Wang (2007); and Challamel and Wang (2008)). There are two nonlocal theories for the vibration of Timoshenko beams, i.e., (1) the Reddy theory (Reddy and Pang, 2008) that allows for length scale coefficient terms in both the normal stress-strain relation and the shear stress-strain relation and (2) the Wang theory (Wang *et al*., 2007) that has the length scale coefficient term in the normal stress-strain relation only. The microstructured beam model developed herein can act as a benchmark solution to evaluate the two aforementioned theories in capturing the nonlocal effect.

Compared with the strong revival of interest in developing nonlocal structural models, the identification of the length scale coefficient $e_0$ in various nonlocal models has not been fully understood. Some experiments have been used to calibrate $e_0$. For instance, Lam *et al*. (2003) provided some experiments in statics for parameter identification of the gradient elasticity model with satisfactory results. Alibert *et al*. (2003) built a generalized gradient elasticity macroscopic continuum from repetitive truss cells. Owing to the complexity of the experiments, numerical approaches including Lattice Dynamics (LD) and Molecular Dynamics (MD) simulations have been used to estimate the length scale coefficient $e_0$ in various nonlocal models. Table I summarizes recent studies on the calibration of $e_0$ from LD to more recent MD simulations. The calibrated $e_0$ ranges from 0 to 19 for different materials, types of problems (wave propagation, vibration, bending and buckling, and defects), boundary conditions and structural geometries. The large spread of $e_0$ values indicates that a direct comparison of $e_0$ among MD/LD simulations data is not possible due to many factors involved; one of them is the different potentials used in the simulations. Although $e_0$ is a key parameter in the nonlocal elasticity theory, there is hitherto no rigorous study being made on estimating it.

The objectives of this study are: (a) to develop analytical exact solutions for free vibration of microstructured beam model with simply supported ends; (b) to develop analytical approximate solutions via Padé approximants, as a general technique, for free vibration of microstructured beam model; (c) to calibrate the length scale coefficient $e_0$ of nonlocal Timoshenko beams by using the proposed exact solutions in (a). Simple analytical expressions of $e_0$ related to geometrical properties and vibration modes are obtained via Padé approximants. Two nonlocal Timoshenko beam models, i.e., the Wang model and the Reddy model, are assessed based on the microstructured beam model by considering the vibration problem of CNTs.

## II. SOLUTIONS FOR FREE VIBRATION OF DISCRETE MICROSTRUCTURED BEAM MODEL

### A. Governing equations

A continuous beam of length $L$ with simply supported ends is discretized into finite rigid segments (or repetitive cells) connected by elastic springs, giving rise to the so-called microstructured beam model. The microstructured beam is composed of $n$ repetitive cells of size denoted by $a$. In other words, the length of the beam $L = n \times a$, i.e., the number of repetitive cells multiplied by the length of each cell. Note that $a$ is the internal characteristic length (which may be interpreted as lattice parameter, C–C bond length, granular size, or even the size of the repetitive frame).

TABLE I. Calibration of length scale coefficient $e_0$ using Lattice Dynamics (LD) and Molecular Dynamics (MD) simulations.

| References | $e_0$ | Physical structures | Type of problems | Calibrated by |
|---|---|---|---|---|
| Eringen (1983) | 0.39 | Aluminum | Wave propagation in solids | LD |
| Eringen (1983) | 0.31 | KCI | Wave propagation in solids | LD |
| Zhang *et al*. (2005) | 0.82 | CNTs | Buckling of shells | MD |
| Wang and Hu (2005) | 0.288 | CNTs | Wave propagation in beams | MD |
| Zhang *et al*. (2006) | 8.79 | Graphene | Defects | MD |
| Duan *et al*. (2007) | 0.0–19.0 | CNTs | Free vibration of beams | MD |
| Hu *et al*. (2008) | 0.2–0.6 | CNTs | Wave propagation in shells | MD |
| Zhang *et al*. (2009) | 1.25 | CNTs | Free vibration of beams | MD |
| Arash and Ansari (2010) | 11.97–14.08 | CNTs | Free vibration of shells | MD |
| Ansari *et al*. (2010) | 6.13–9.93 | Graphene | Free vibration of plates | MD |

As shown in Figure 1, consider two repetitive cell members, i.e., $i$ and $i+1$, which are isolated from the microstructured beam. Three nodes $i$, $i+1$, and $i+2$ are connected with these two members. In the undeformed state, the members are parallel to axis $x$ of the Cartesian axes $x$ and $y$. The repetitive cells are connected by elastic rotational and transverse springs at their ends, which have the stiffnesses $C$ and $S$, respectively. The stiffnesses $C$ and $S$ are given by $C = nEI/L = EI/a$ and $S = nK_sGA/L = K_sGA/a$ where $E$ is Young's modulus, $G$ the shear modulus, $K_s$ the shear correction factor to compensate for the error in assuming a constant shear strain/stress over the beam thickness in the Timoshenko beam theory, $I$ the second moment of area, and $A$ the cross-sectional area of the beam. The total mass of the microstructured beam is lumped at the internal nodes as $m_i = m = \rho AL/n = \rho Aa$ for $i = 2, 3,\ldots, n$, where $\rho$ is the mass density, and $m_i = \rho AL/2n = \rho Aa/2$ for the two end nodes ($i = 1$ and $i = n+1$) since the end nodes have only one rigid segment contributing to the nodal mass.

Let the members have transverse displacements $w_i$ and $w_{i+1}$ and rotation $\theta_i$ and $\theta_{i+1}$ on nodes $i$ and $i+1$, respectively. One can get the displacement $(w_{i+1} - w_i - a\theta_i)$ on the shear spring and rotation $(\theta_{i+1} - \theta_i)$ on the rotational spring at node $i+1$. Thus, the total elastic potential $U_b$ of the deformed rotational springs in the beam is given by

$$U_b = \frac{1}{2}\sum_{i=1}^{n} C(\theta_{i+1} - \theta_i)^2. \tag{1}$$

The total elastic potential $U_s$ of the deformed shear spring energy is given by

$$U_s = \frac{1}{2}\sum_{i=1}^{n+1} S(w_{i+1} - w_i - a\theta_i)^2. \tag{2}$$

The total kinetic energy $T$ is given by

$$T = \frac{1}{2}\sum_{i=1}^{n+1} m\dot{w}_i^2 + \frac{1}{2}\sum_{i=1}^{n} I_m\dot{\theta}_i^2, \tag{3}$$

where $I_m = \rho IL/n = \rho Ia$ is the second moment of inertia of the beam segment.

The Lagrangian $L$ of the system is defined as: $L = T - U_b - U_s$, which is given by (Ostoja-Starzewski, 2002)

$$L = \frac{1}{2}\sum_{i=1}^{n+1} m\dot{w}_i^2 + \frac{1}{2}\sum_{i=1}^{n} I_m\dot{\theta}_i^2 - \frac{1}{2}\sum_{i=1}^{n+1} S(w_{i+1} - w_i - a\theta_i)^2 - \frac{1}{2}\sum_{i=1}^{n} C(\theta_{i+1} - \theta_i)^2. \tag{4}$$

The Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial w_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}_j}\right), \tag{5}$$

$$\frac{\partial L}{\partial \theta_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_j}\right). \tag{6}$$

Note that the differentiation of finite difference functions can be conducted as

$$\frac{\partial U_b}{\partial \theta_j} = \sum_{i-1}^{n-1} C(\theta_{i+1} - \theta_i)(\delta_{i,j+1} - \delta_{i,j}) = C(\theta_{j+1} + \theta_{j-1} - 2\theta_j)\ , \tag{7}$$

$$\frac{\partial U_s}{\partial \theta_j} = \sum_{i-1}^{n-1} S(w_{i+1} - w_i - a\theta_i)(\delta_{i,j+1} - \delta_{i,j}) = S(w_{j+1} + w_{j-1} - 2w_j) - aS(\theta_j - \theta_{j-1})\ , \tag{8}$$

where $\delta_{i,j}$ is the Kronecker function and $2 \le j \le n$. Noting that for the sake of simplicity, the index $j$ in Eqs. (7) and (8) is changed back to $i$ in the derivations below.

By combining the Lagrangian $L$ in Eq. (4) with Eqs. (5) and (6), one obtains

$$S(w_{i+1} + w_{i-1} - 2w_i) - aS(\theta_i - \theta_{i-1}) = m\ddot{w}_i, \tag{9}$$

$$C(\theta_{i+1} + \theta_{i-1} - 2\theta_i) + aS(w_{i+1} - w_i - a\theta_i) = I_m\ddot{\theta}_i. \tag{10}$$

By denoting $L_i = M_{i+1} + M_{i-1} - 2M_i$, $M_i = w_{i+1} + w_{i-1} - 2w_i$, $N_i = \theta_i - \theta_{i-1}$, Eq. (9) can be written compactly as

$$aSN_i - SM_i + m_i\ddot{w}_i = 0. \tag{11}$$

By shifting the index of Eq. (10) by one, i.e., from $i$ to $i-1$,

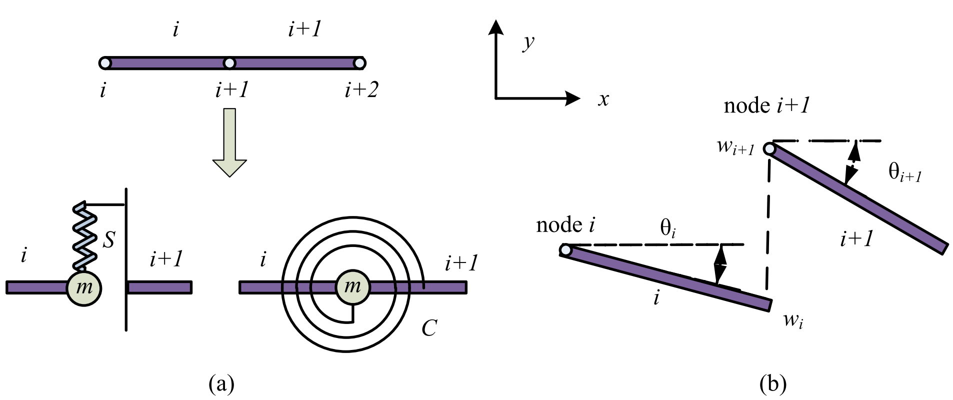


FIG. 1. Two members of a microstructured beam (a) undeformed state with two springs on each node and (b) deformed state on node $i+1$, assume rotation of the member $i+1$ is around the node $i+1$.

$$C(\theta_i+\theta_{i-2}-2\theta_{i-1})+aS(w_i-w_{i-1}-a\theta_{i-1})=I_m\ddot{\theta}_{i-1}. \tag{12}$$

The subtraction of Eq. (10) from Eq. (12) yields

$$C(N_{i+1}+N_{i-1}-2N_i)+aS(M_i-aN_i)=I_m\ddot{N}_i. \tag{13}$$

Eliminating $N$ from Eq. (13) by using Eq. (11) leads to

$$\frac{C}{a}L_i-\left(\frac{Cm}{aS}+\frac{I_m}{a}\right)\ddot{M}_i+am\ddot{w}_i+\frac{mI_m}{aS}\ddddot{w}_i=0, \tag{14}$$

where $2\le i\le n$.

Assuming harmonic motion $w_i=\bar{w}_ie^{-i\omega t}$, Eq. (14) yields

$$\bar{L}_i+\omega^2\left(\frac{m}{S}+\frac{I_m}{C}\right)\bar{M}_i-\omega^2\left(\frac{a^2m}{C}-\frac{mI_m\omega^2}{SC}\right)\bar{w}_i=0. \tag{15}$$

By introducing the following non-dimensional parameters:

$$\lambda^2=\frac{\omega^2\rho AL^4}{EI},\quad \mu_s=\frac{E}{K_sG},\quad \zeta=\frac{L\sqrt{A}}{\sqrt{I}},\quad w_n=\frac{\bar{w}}{L},\quad \eta=\frac{x}{L}. \tag{16}$$

Equation (15) can be written as

$$\bar{L}_i+\frac{\lambda^2}{n^2\zeta^2}(1+\mu_s)\bar{M}_i+\frac{\lambda^2}{n^4}\left(\frac{\mu_s\lambda^2}{\zeta^4}-1\right)\bar{w}_i=0, \tag{17}$$

or

$$\begin{aligned}&(\bar{w}_{i+2}-4\bar{w}_{i+1}+6\bar{w}_i-4\bar{w}_{i-1}+\bar{w}_{i-2})\\&+\frac{\lambda^2}{n^2\zeta^2}(1+\mu_s)(\bar{w}_{i+1}-2\bar{w}_i+\bar{w}_{i-1})\\&+\frac{\lambda^2}{n^4}\left(\frac{\mu_s\lambda^2}{\zeta^4}-1\right)\bar{w}_i=0,\end{aligned} \tag{18}$$

where $2\le i\le n$. For the simply supported ends, the bending moments on the two end nodes $i=1$ and $i=n+1$ are zero leading to

$$\bar{w}_1=0,\ \ \bar{w}_n=0,\quad \bar{w}_0=-\bar{w}_2,\quad \bar{w}_{n+2}=-\bar{w}_n. \tag{19}$$

When $n\to\infty$,

$$\bar{L}_i=\frac{a^4}{L^4}\frac{d^4w_n}{d\eta^4},\quad \bar{M}_i=\frac{a^2}{L^2}\frac{d^2w_n}{d\eta^2}, \tag{20}$$

and Eq. (17) reduces to

$$\frac{d^4w_n}{d\eta^4}+\frac{\lambda^2}{\zeta^2}(1+\mu_s)\frac{d^2w_n}{d\eta^2}+\lambda^2\left(\frac{\mu_s\lambda^2}{\zeta^4}-1\right)w_n=0, \tag{21}$$

which is the governing equation for the vibration of local Timoshenko beams. Together with $\zeta\to\infty$, Eq. (21) further reduces to the governing equation for the vibration of local (or classical) Euler–Bernoulli beams.

### B. Exact solutions via matrix decomposition

The combination of Eqs. (17) and (19) yields, in a matrix form,

$$\left[[L]+\frac{\lambda^2}{n^2\zeta^2}(1+\mu_s)[M]+\frac{\lambda^2}{n^4}\left(\frac{\mu_s\lambda^2}{\zeta^4}-1\right)[I][\bar{w}]\right]=0, \tag{22}$$

where $[I]$ is identity matrix, $[\bar{w}]=[\bar{w}_2,\ \bar{w}_3,\cdots,\ \bar{w}_n]^T$ is a $(n-1)$ vector, and $[L]$ and $[M]$ are $(n-1)\times(n-1)$ matrices given by

$$[L]=\begin{bmatrix}5&-4&1&\cdots&0\\-4&6&-4&1&\\1&-4&\ddots&\ddots&\vdots\\0&1&\cdots&6&-4\\&&\cdots&-4&5\end{bmatrix}, \tag{23}$$

and

$$[M]=\begin{bmatrix}-2&1&\cdots&\cdots&0\\1&-2&\cdots&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&\cdots&-2&1\\0&\cdots&\cdots&1&-2\end{bmatrix}. \tag{24}$$

Noting that

$$[L]=[M]^2. \tag{25}$$

Thus, Eq. (22) can be written as

$$[[M]-A_1[I]][\bar{w}]=0, \tag{26}$$

where

$$A_1=\frac{-B_1\pm\sqrt{B_1^2-4B_2}}{2}, \tag{27}$$

and

$$B_1=\frac{\lambda^2}{n^2\zeta^2}(1+\mu_s),\ \ B_2=\frac{\lambda^2}{n^4}\left(\frac{u_s\lambda^2}{\zeta^4}-1\right). \tag{28}$$

Let the determinant of the coefficient matrices in Eq. (26) be zero, i.e.,

$$|[M]-A_1[I]|=0. \tag{29}$$

By taking the diagonal elements "$-2$" in the matrix $[M]$ into the identity matrix, one obtains

$$|[M]^*-(A_1+2)[I]|=0, \tag{30}$$

where

$$[M]^*=\begin{bmatrix}0&1&\cdots&\cdots&0\\1&0&\cdots&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&\cdots&0&1\\0&\cdots&\cdots&1&0\end{bmatrix}. \tag{31}$$

Now, the relation between the matrix $[M]^*$ and Chebyshev polynomials will be established. The second kind Chebyshev polynomial is given by (Abramowitz and Stegun, 1970)

$$U_n(x) = (n+1)_2F_1\left(-n,\ n+2,\ \frac{3}{2},\ \frac{1}{2}[1-x]\right), \quad (32)$$

where ${}_2F_1([a,\ b],\ [c],\ x)$ is a hypergeometric function and the series form is given by

$${}_2F_1([a,b],[c],x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n. \quad (33)$$

The hypergeometric function has earlier been adopted to solve the transverse vibration problem of non-uniform annular plates (Duan *et al.*, 2005), the lateral-torsional buckling problem of linearly tapered cantilever (Challamel *et al.*, 2007), the axisymmetric bending problem of micro/nanoscale circular plates (Duan and Wang, 2007), the buckling problem of columns with allowance for self-weight (Duan and Wang, 2008), and axisymmetric transverse vibration problem of circular cylindrical shells with variable thickness (Duan and Koh, 2008), the flexural-torsional buckling problem of cantilever (Challamel *et al.*, 2010), and the lateral-torsional stability boundaries for polygonally depth-tapered strip cantilevers (Andrade *et al.*, 2012).

The Frobenius companion matrix of a second kind Chebyshev polynomial $U_{n-1}(x/2)$ is given by (Horn and Johnson, 1990)

$$[M]^c = \begin{bmatrix} 0 & 0 & \cdots & \cdots & F_0 \\ 1 & 0 & \cdots & \cdots & F_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & F_{n-2} \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}, \quad (34)$$

where $F_i (i = 1..n-2)$ are the coefficients of the hypergeometric function ${}_2F_1([a,b],[c],x)$.

Linear Algebra manipulations show that the matrices $[M]^*$ and $[M]^c$ are similar so that they have the same characteristic polynomial, and consequently the same eigenvalues, i.e., they have the same roots as the second kind Chebyshev polynomial $U_{n-1}(x/2)$, which are given by

$$x_k = 2\cos\left(\frac{k\pi}{n}\right), \quad k = 1,\ \ldots,\ n-1. \quad (35)$$

Therefore

$$A_1 + 2 = 2\cos\left(\frac{k\pi}{n}\right), \quad k = 1,\ \ldots,\ n-1, \quad (36)$$

and

$$A_1 = \frac{-B_1 \pm \sqrt{B_1^2 - 4B_2}}{2} = -2\left(1 - \cos\left(\frac{k\pi}{n}\right)\right)$$
$$= -4\sin^2\left(\frac{k\pi}{2n}\right) = -\delta. \quad (37)$$

Equation (37) can be written as

$$\pm\sqrt{B_1^2 - 4B_2} = B_1 - 2\delta, \quad (38)$$

or

$$B_1^2 - 4B_2 = (B_1 - 2\delta)^2. \quad (39)$$

By substituting Eq. (27) into Eq. (39), one obtains

$$\frac{\mu_s}{n^4\zeta^4}\lambda^4 - \left(\frac{\delta}{n^2\zeta^2}(1+\mu_s) + \frac{1}{n^4}\right)\lambda^2 + \delta^2 = 0. \quad (40)$$

When $n \to \infty$, $\delta = 4\sin^2(k\pi/2n) = k^2\pi^2/n^2$, Eq. (40) becomes the characteristic equation for the vibration of local Timoshenko beams

$$\frac{\mu_s}{\zeta^4}\lambda^4 - \left(\frac{k^2\pi^2}{\zeta^2}(1+\mu_s) + 1\right)\lambda^2 + k^4\pi^4 = 0. \quad (41)$$

When $\zeta \to \infty$, Eq. (40) becomes the characteristic equation for the vibration of discrete beam elements with rotational springs, as obtained by Santoro and Elishakoff (2006)

$$\frac{\lambda}{k^2\pi^2} = \frac{\delta n^2}{k^2\pi^2} = \frac{\sin^2\left(\frac{k\pi}{2n}\right)}{\left(\frac{k\pi}{2n}\right)^2}. \quad (42)$$

For the case when both $n \to \infty$ and $\zeta \to \infty$, Eq. (41) will further reduce to the characteristic equation for the vibration of local (or classical) Euler–Bernoulli beams and leads to the classical solution where $\lambda = k^2\pi^2$.

Here, two non-dimensional parameters $\alpha = k^2\pi^2/n^2$ and $\beta = k^2\pi^2/\zeta^2$ are defined. For a given vibration mode number $k$, $\alpha$ represents the number of repetitive cells, while $\beta$ represents the slenderness ratio $\zeta$. Assume a beam with a circular hollow section

$$A = \pi dt, \quad I = \frac{\pi d^3 t}{8}, \quad \zeta^2 = \frac{L^2 A}{I} = 8\left(\frac{L}{d}\right)^2, \quad K_s = \frac{2(1+\mu)}{4+3\mu}, \quad (43)$$

where $d$ and $t$ are the diameter and thickness of the beam section. The Poisson ratio is assumed to be $\mu = 0.19$ which is a typical value for CNTs. Thus $\alpha$ and $\beta$ can be written as

$$\alpha = \frac{k^2\pi^2}{n^2} = \frac{\pi^2}{(n/k)^2}, \quad (44)$$

$$\beta = \frac{k^2\pi^2}{\zeta^2} = \frac{\pi^2}{(L/k)^2}\frac{I}{A} = \frac{8\pi^2}{(L/(dk))^2}. \quad (45)$$

As far as beam vibration is concerned, assume $n/k \geq 10$ and $L/(dk) \geq 10$ (Timoshenko *et al.*, 1974), the ranges of $\alpha$ and $\beta$ can be identified as $0 < \alpha \leq 0.1$ and $0 < \beta \leq 0.5$.

From Eq. (40), one gets

$$\frac{\lambda_E^2}{k^4\pi^4} = \left(1 + \frac{1}{\mu_s}\right)\frac{\delta}{2\alpha\beta} + \frac{1}{2\mu_s\beta^2}\left(1 \pm \sqrt{(\mu_s - 1)^2\frac{\delta^2\beta^2}{\alpha^2} + 2(\mu_s + 1)\frac{\delta\beta}{\alpha} + 1}\right), \quad (46)$$

where $\lambda_E$ denotes the exact frequency obtained from Eq. (40). Maclaurin series expansion (Taylor series expansion centered at zero) with respect to $\alpha$ on $\delta$ furnishes

$$\delta = 4\sin^2\left(\frac{k\pi}{2n}\right) = 4\sin^2\left(\frac{\sqrt{\alpha}}{2}\right) = \alpha\left(1 - \frac{1}{12}\alpha + \frac{1}{360}\alpha^2\right) + O(\alpha^4). \quad (47)$$

The higher root of Eq. (46) can be expanded as

$$\frac{\lambda_E^2}{k^4\pi^4} = \frac{(1+\mu_s)\beta + 1}{\mu_s\beta^2} - 1 + \frac{1}{6}\alpha + (1+\mu_s)\beta + O(\alpha^2,\ \alpha\beta,\ \beta^2). \quad (48)$$

When $\beta \to 0$ and $\alpha \to 0$ corresponding to low frequency vibration, it leads to a singularity on $\lambda$ in Eq. (48). Thus, the higher root is abandoned.

By substituting Eq. (47) into the lower root of Eq. (46) and then by applying Maclaurin series expansion with respect to $\alpha$ and $\beta$, one obtains

$$\begin{aligned}\frac{\lambda_E^2}{k^4\pi^4} &= 1 - (1+\mu_s)\beta + (1+3\mu_s+\mu_s^2)\beta^2 \\ &+ \left(-\frac{1}{6} + \frac{1}{4}(1+\mu_s)\beta - \frac{1}{3}(1+3\mu_s+\mu_s^2)\beta^2\right)\alpha \\ &+ \left(\frac{1}{80} - \frac{7}{240}(1+\mu_s)\beta + \frac{19}{360}(1+3\mu_s+\mu_s^2)\beta^2\right)\alpha^2 \\ &+ O(\alpha^3,\ \beta^3). \end{aligned} \quad (49)$$

For the case when $\beta \to 0$ and $\alpha \to 0$, i.e., the beam with a large number of repetitive cells undergoing a low frequency vibration, Eq. (49) reduces to simply $\lambda = k^2\pi^2$, which is the well-known frequency solution for local (or classical) Euler–Bernoulli beams with simply supported ends. It is worth noting that microstructured beams have lower frequencies when compared with their corresponding continuum beams owing to the negative coefficients of $\alpha$ and $\beta$ in Eq. (49), i.e., $-1/6$, and $-(1+\mu_s)$, respectively.

### C. Approximate solutions via continualization

The condition in Eq. (25) may be only valid for simply supported boundary conditions. The exact solutions, as shown in Eq. (46), may not be admissible for other boundary conditions such as clamped or free boundary conditions. It is thus desirable to have a general technique for solving the discrete microstructured problems. For example, one may resort to continualization via Padé approximants. The Padé approximants often give a better approximation of a function than its Taylor series counterpart and the detailed definition can be found in Baker and Graves-Morris (1996). One of the calculation methods for a Padé approximant of a function is given here. The Maclaurin series expansion of a function $F(x)$ is given by

$$F(x) = \sum_{k=0}^{\infty} a_k x^k. \quad (50)$$

The Padé approximant $[L/M]$ of the function $F(x)$ is given by

$$F(x) \approx \frac{p_0 + p_1 x + \cdots + p_L x^L}{1 + q_1 x + \cdots + q_M x^M}, \quad (51)$$

where the $p_0,\ p_1, \ldots, p_L$ and $q_1,\ q_2, \ldots, q_M$ can be obtained by solving

$$\begin{gathered} a_0 = p_0 \\ a_1 + a_0 q_1 = p_1 \\ \cdots \\ a_L + a_{L-1}q_1 + \cdots + a_0 q_L = p_L \\ a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ \cdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0. \end{gathered} \quad (52)$$

Now the so-called pseudo differential operators in bridging the discrete and the equivalent continuous system are introduced here. The following relation between the discrete and the equivalent continuous system holds for a sufficiently smooth displacement function (Andrianov *et al.*, 2010 and Challamel *et al.*, 2013)

$$\begin{aligned} \bar{w}_i &= \bar{w}(x), \\ \bar{w}_{i+1} &= \sum_{k=0}^{\infty} \frac{a^k D_x^k}{k!} \bar{w}(x) = \exp(aD_x)\bar{w}(x), \end{aligned} \quad (53)$$

where $x = ia$, $\bar{w}_i$ and $\bar{w}(x)$ are the displacements in the discrete and the equivalent continuous system, respectively. $D_x = \partial/\partial x$ and $\exp(aD_x)$ belongs to the pseudo differential operators.

In view of Eq. (53), $\bar{L}_i = \bar{M}_{i+1} + \bar{M}_{i-1} - 2\bar{M}_i$ and $\bar{M}_i = \bar{w}_{i+1} + \bar{w}_{i-1} - 2\bar{w}_i$ can be written as

$$\begin{aligned} \bar{L}_i &= \bar{w}_{i+2} - 4\bar{w}_{i+1} + 6\bar{w}_i - 4\bar{w}_{i-1} + \bar{w}_{i-2} \\ &= (\exp(2aD_x) - 4\exp(aD_x) + 6 - 4\exp(-aD_x) \\ &\quad + \exp(-2aD_x))\bar{w}(x) \\ &= \left(1 + \frac{1}{6}D_x^2 + \frac{1}{80}D_x^4 + O(D_x^6)\right) a^4 D_x^4 \bar{w}(x), \end{aligned} \quad (54)$$

and

$$\begin{aligned} \bar{M}_i &= \bar{w}_{i+1} - 2\bar{w}_i + \bar{w}_{i-1} \\ &= (\exp(aD_x) - 2 + \exp(-aD_x))\bar{w}(x) \\ &= \left(1 + \frac{1}{12}D_x^2 + \frac{1}{360}D_x^4 + O(D_x^6)\right) a^2 D_x^2 \bar{w}(x). \end{aligned} \quad (55)$$

The Padé approximant [1/4] of Eq. (54) yields

$$\begin{aligned} \bar{L}_i &= \left(1 + \frac{1}{6}D_x^2 + \frac{1}{80}D_x^4 + O(D_x^6)\right) a^4 D_x^4 \bar{w}(x) \\ &= \frac{1}{1 - \frac{1}{6}D_x^2 + \frac{11}{720}D_x^4} a^4 D_x^4 \bar{w}(x) \\ &= \frac{1}{\left(1 - \frac{1}{12}D_x^2\right)^2 + \frac{1}{120}D_x^4} a^4 D_x^4 \bar{w}(x) \\ &\approx \frac{1}{\left(1 - \frac{1}{12}D_x^2\right)^2} a^4 D_x^4 \bar{w}(x). \end{aligned} \quad (56)$$

The Padé approximant [1/2] of Eq. (55) yields

$$\bar{M}_i = \left(1 + \frac{1}{12}D_x^2 + \frac{1}{360}D_x^4 + O(D_x^6)\right)a^2 D_x^2 \bar{w}(x) = \frac{1}{1 - \frac{1}{12}D_x^2} a^2 D_x^2 \bar{w}(x). \tag{57}$$

In view of Eqs. (56) and (57) and non-dimensional parameters in Eq. (16), Eq. (15) can be written as:

$$\left(\frac{\lambda^2}{144n^4}\left(\frac{\mu_s\lambda^2}{\zeta^4} - 1\right) - \frac{\lambda^2}{12\zeta^2 n^2}(\mu_s + 1) + 1\right)\frac{d^4 w_n}{d\eta^4} + \lambda^2\left(\frac{1}{6n^2} + \frac{1}{\zeta^2}(\mu_s + 1) - \frac{\mu_s\lambda^2}{6\zeta^4 n^2}\right)\frac{d^2 w_n}{d\eta^2} + \lambda^2\left(\frac{\mu_s\lambda^2}{\zeta^4} - 1\right)w_n(\eta) = 0. \tag{58}$$

For beams with simply supported ends, the solution for Eq. (58) takes the form of

$$w_n(\eta) = \sin(k\pi\eta), \tag{59}$$

where $k$ is the vibration mode number. The combination of Eq. (59) and Eq. (58) yields

$$\left(\frac{k^4\pi^4\mu_s}{144\zeta^4 n^4} + \frac{k^2\pi^2\mu_s}{6\zeta^4 n^2} + \frac{\mu_s}{\zeta^4}\right)\lambda^4 - \left(\frac{k^4\pi^4}{144n^4} + \left(\frac{k^2\pi^2}{6} + \frac{k^4\pi^4}{12\zeta^2}(1 + \mu_s)\right)\frac{1}{n^2} + (1 + \mu_s)\frac{k^2\pi^2}{\zeta^2} + 1\right)\lambda^2 + k^4\pi^4 = 0. \tag{60}$$

From Eq. (60) and considering the non-dimensional parameters $\alpha = k^2\pi^2/n^2$ and $\beta = k^2\pi^2/\zeta^2$, one obtains

$$\frac{\lambda_P^2}{k^4\pi^4} = \frac{6(\mu_s+1)}{\mu_s(\alpha+12)\beta} + \frac{1}{2\mu_s\beta^2} \times \left(1 - \sqrt{1 + \frac{(\mu_s+1)(24\alpha+288)\beta + 144(\mu_s-1)^2\beta^2}{(\alpha+12)^2}}\right), \tag{61}$$

where $\lambda_P$ denotes the frequency obtained from Eq. (60) for the equivalent continuous counterpart of a discrete system via Padé approximants.

When $\beta \to 0$ and $\alpha \to 0$, by taking Maclaurin series expansion of Eq. (61) with respect to $\alpha$ and $\beta$, one obtains

$$\frac{\lambda_P^2}{k^4\pi^4} = 1 - (1 + \mu_s)\beta + (1 + 3\mu_s + \mu_s^2)\beta^2 + \left(-\frac{1}{6} + \frac{1}{4}(1 + \mu_s)\beta - \frac{1}{3}(1 + 3\mu_s + \mu_s^2)\beta^2\right)\alpha + \left(\frac{1}{48} - \frac{1}{24}(1 + \mu_s) + \frac{5}{72}(1 + 3\mu_s + \mu_s^2)\beta^2\right)\alpha^2 + O(\alpha^3,\ \beta^3). \tag{62}$$

Comparing Eq. (49) with Eq. (62), the difference is in the order of $\alpha^2$ indicating Padé approximant solutions have the first order accuracy. Assuming $\mu = 0.19$ and taking the beam of circular cross section as indicated in Eq. (43), one gets $\mu_s = 4.57$. For $\alpha = 0.1$ and $\beta = 0.5$, the difference between $\lambda_P$ and $\lambda_E$ is $(\lambda_P/\lambda_E - 1) = 5.484 \times 10^{-5}$.

## III. NONLOCAL TIMOSHENKO BEAM MODELS AND CALIBRATION OF LENGTH SCALE COEFFICIENT

### A. Nonlocal Timoshenko beam models

As mentioned earlier, there are two proposed nonlocal theories for the vibration of Timoshenko beams, i.e., the Reddy theory (Reddy and Pang, 2008) and the Wang theory (Wang *et al.*, 2007) where in the latter theory the length scale coefficient term appears in the normal stress-strain relation only. Microstructured beam model can act as a benchmark solution to compare the two aforementioned theories in capturing the nonlocal effect. The governing equations for both theories are briefly summarized here. The motion of equations are given by

$$\frac{dM}{dx} = Q - \rho I\omega^2\phi, \tag{63}$$

$$\frac{dQ}{dx} = -\rho A\omega^2 w, \tag{64}$$

where $x$ is the longitudinal coordinate measured from the left end of the beam, $w$ the transverse displacement, $\phi$ the rotation due to bending, $M$ the bending moment, and $Q$ shear force.

For an elastic material in the one dimensional case, the nonlocal constitutive relations are given by

$$M - (e_0 a)^2\frac{d^2M}{dx^2} = EI\frac{d\phi}{dx}, \tag{65}$$

$$Q - R_s(e_0 a)^2\frac{d^2Q}{dx^2} = K_s GA\left(\phi + \frac{dw}{dx}\right), \tag{66}$$

where a is the internal characteristic length (defined in Sec. II A), $e_0$ the length scale coefficient, the scalar indicator $R_s = 0$ for the Wang model (Wang *et al.*, 2007) and $R_s = 1$ for the Reddy model (Reddy and Pang, 2008).

In view of Eq. (16) and by eliminating $\phi$ from Eq. (63) to Eq. (66), one obtains

$$\left(1 - \frac{e_0^2\lambda^2}{\zeta^2 n^2}(1 + R_s\mu_s) + \frac{e_0^4 R_s\mu_s\lambda^4}{\zeta^4 n^4}\right)\frac{d^4 w_n}{d\eta^4} + \lambda^2\left(\frac{e_0^2}{n^2} + \frac{1}{\zeta^2}(1 + \mu_s) - (1 + R_s)\frac{e_0^2\mu_s\lambda^2}{\zeta^4 n^2}\right)\frac{d^2 w_n}{d\eta^2} + \lambda^2\left(\frac{\mu_s\lambda^2}{\zeta^4} - 1\right)w_n(\eta) = 0. \tag{67}$$

Again, the solutions of Eq. (67) take the form of Eq. (59). Combining them yields

$$\left(\frac{R_s k^4 \pi^4 \mu_s e_0^4}{\zeta^4 n^4} + \frac{(1+R_s) k^2 \pi^2 \mu_s e_0^2}{\zeta^4 n^2} + \frac{\mu_s}{\zeta^4}\right)\lambda^4 - \left(1 + (1+\mu_s)\frac{k^2\pi^2}{\zeta^2} + \left(k^2\pi^2 + (1+R_s\mu_s)\frac{k^4\pi^4}{\zeta^2}\right)\frac{e_0^2}{n^2}\right)\lambda^2 + k^4\pi^4 = 0. \tag{68}$$

Consider two non-dimensional parameters $\alpha = k^2\pi^2/n^2$ and $\beta = k^2\pi^2/\zeta^2$, the frequency $\lambda$ is given by the lower real root of Eq. (68)

$$\frac{\lambda_N^2}{k^4\pi^4} = \frac{((1+R_s\mu_s)\beta + 1)\alpha e_0^2 + (1+\mu_s)\beta + 1 - \sqrt{\Delta}}{2\mu_s(1 + (1+R_s)\alpha e_0^2 + R_s\alpha^2 e_0^4)\beta^2}, \tag{69}$$

where

$$\Delta = (((R_s\mu_s - 1)\beta + 1)^2 + 4\beta)\alpha^2 e_0^4 + 2((\mu_s - 1)(R_s\mu_s - 1)\beta^2 + (2 + (R_s+1)\mu_s)\beta + 1)\alpha e_0^2 + (\mu_s - 1)^2\beta^2 + 2(\mu_s + 1)\beta + 1.$$

### B. Calibration of $e_0$ in the ranges of $0 < \alpha \le 0.1$ and $0 < \beta \le 0.5$

The calibration of $e_0$ is conducted by using the exact solutions (Eq. (46)) of the microstructured beam model. The substitution of frequency $\lambda_{Ek} = \lambda_E/(k^2\pi^2) = \lambda_N/(k^2\pi^2)$ from Eq. (46) into Eq. (69) yields

$$e_0^2 = \frac{1 + (R_s\mu_s + 1)\beta - \lambda_{Ek}^2\mu_s(R_s+1)\beta^2}{2R_s\mu_s\beta^2\lambda_{Ek}^2\alpha} + \frac{\sqrt{(\lambda_{Ek}^2\mu_s(R_s-1)\beta^2 - (R_s\mu_s - 1)\beta + 1)^2 + 4R_s\mu_s\beta}}{2R_s\mu_s\beta^2\lambda_{Ek}^2\alpha}. \tag{70}$$

It can be seen that the length scale coefficient $e_0$ is determined by $\alpha$, $\beta$, and $\mu_s$, which are related with structural geometry, vibration mode number, and shear stiffness. In order to compare length scale coefficients $e_0^W$ and $e_0^R$ within the ranges of $0 < \alpha \le 0.1$ and $0 < \beta \le 0.5$, where $e_0^W$ and $e_0^R$ are obtained from Eq. (70) with $R_s = 0$ and $R_s = 1$, respectively, the same circular hollow section beam as shown in Eq. (43) is adopted.

Figure 2 shows that $e_0^W$ and $e_0^R$ are varied about 0.40 to 0.54 and 0.40 to 0.34, respectively. These ranges are in agreement with $e_0 = 0.408$ obtained by Challamel *et al.* (2013) for the free vibration of a microstructured model with rotational springs only. In general, the Wang model predicts a larger length scale coefficient when compared with the Reddy model, especially when $\alpha$ and $\beta$ are relatively large, i.e., for short beams with higher frequency vibrations. More detailed evaluation on the predication of length scale coefficient $e_0$ from the Wang model and the Reddy model will be presented in Sec. IV.

Equation (70) is not convenient for practical use. So a Padé approximant of $e_0$ is developed as follows. By taking Maclaurin series expansion of Eq. (70) with respect to $\alpha$ and $\beta$, one obtains

$$e_0^2 = \frac{1}{6} - \frac{1}{12}((2R_s - 1)\mu_s + 1)\beta + \frac{1}{12}((2R_s^2 - R_s)\mu_s^2 + (5R_s - 1)\mu_s + 1)\beta^2 + \frac{1}{720}(11 - ((11R_s - 3)\mu_s + 8)\beta + ((11R_s^2 - 3R_s)\mu_s^2 + (30R_s - 3)\mu_s + 8)\beta^2)\alpha. \tag{71}$$

As it can be seen from Figure 2, the variation of $e_0$ along $\alpha$ is not significant and thus $\alpha$ can be ignored from Eq. (71).

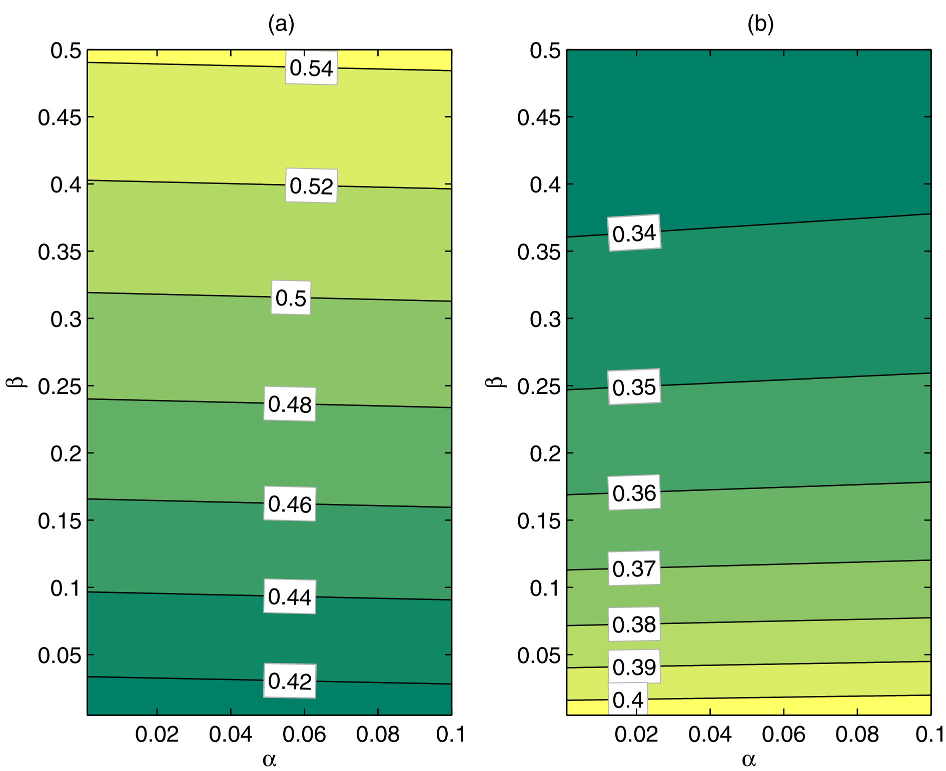


FIG. 2. Contour of length scale coefficient within the ranges of $0 < \alpha \le 0.1$ and $0 < \beta \le 0.5$: (a) $e_0^W$ and (b) $e_0^R$.

Following the procedure outline in Eq. (52), the Padé approximant [1/1] of $e_0$ with respect to $\beta$ is given by

$$e_0^2=\frac{1}{12}\frac{((2R_s-1)\mu_s^2+6R_s\mu_s+1)\beta+2(2R_s-1)\mu_s+2}{((2R_s^2-R_s)\mu_s^2+(5R_s-1)\mu_s+1)\beta+(2R_s-1)\mu_s+1}. \tag{72}$$

For the Wang model, $R_s=0$

$$e_0^2=\frac{1}{12}\frac{(\mu_s+1)\beta+2}{\beta+1}, \tag{73}$$

and for the Reddy model, $R_s=1$

$$e_0^2=\frac{1}{12}\frac{(\mu_s^2+6\mu_s+1)\beta+2(\mu_s+1)}{(\mu_s^2+4\mu_s+1)\beta+\mu_s+1}. \tag{74}$$

When $\beta\to 0$, the beam has a low frequency vibration and the vibration behavior is more like the Euler–Bernoulli beam. For this case, both the Wang model and the Reddy model give

$$e_0^2=\frac{1}{6}. \tag{75}$$

In the case when $\beta=0.5$, where the beam undergoes a high frequency vibration and the shear effect is strong, the Wang model gives

$$e_0^2=\frac{1}{36}(\mu_s+5), \tag{76}$$

whereas the Reddy model gives

$$e_0^2=\frac{1}{12}\frac{\mu_s^2+10\mu_s+5}{\mu_s^2+6\mu_s+3}>\frac{1}{12}. \tag{77}$$

In order to access the accuracy of the approximated $e_0$ in Eq. (72) when compared with the exact $e_0$ in Eq. (70), Figure 3 shows the comparison between $e_0^{WF}$ and $e_0^{RF}$, which are obtained based on Eq. (72) for the Wang model and the Reddy model, respectively. It is clear that the Padé approximant [1/1] of $e_0$ are very accurate. Within the ranges of $0<\alpha\leq 0.1$ and $0<\beta\leq 0.5$, the difference of $e_0$ is smaller than 5% for the Wang model and 2.5% for the Reddy model.

### C. Calibration of $e_0$ for CNTs

For the CNTs with the geometrical properties in Eq. (43), Table II lists the $e_0$ obtained from Eq. (72) for the vibration of a carbon nanotube with different aspect ratios for the vibration mode $k=1$. For an aspect ratio $L/d=20$, the Wang model gives $e_0=0.409$ while the Reddy model gives $e_0=0.407$. Both models furnish almost the same value of $e_0$. For a shorter CNT with an aspect ratio $L/d=12$, $e_0=0.411$ is for the Wang model and $e_0=0.404$ for the Reddy model. The difference between the Wang model and the Reddy model is 1.73%. It is clear that for a sufficiently long beam, say, the aspect ratio $L/d=20$, the Wang model and the Reddy model produce almost the same length scale coefficient $e_0$ for the lower frequency vibration. With shorter beams, the Reddy model predicts a smaller value of $e_0$, which indicates that the Reddy model is closer to the microstructured beam model. For free vibration of simply supported CNTs with different aspect ratios ranging from 12 to 20, $e_0$ is about 0.4. Interestingly, the current prediction of $e_0$ based on microstructured beam model can produce a

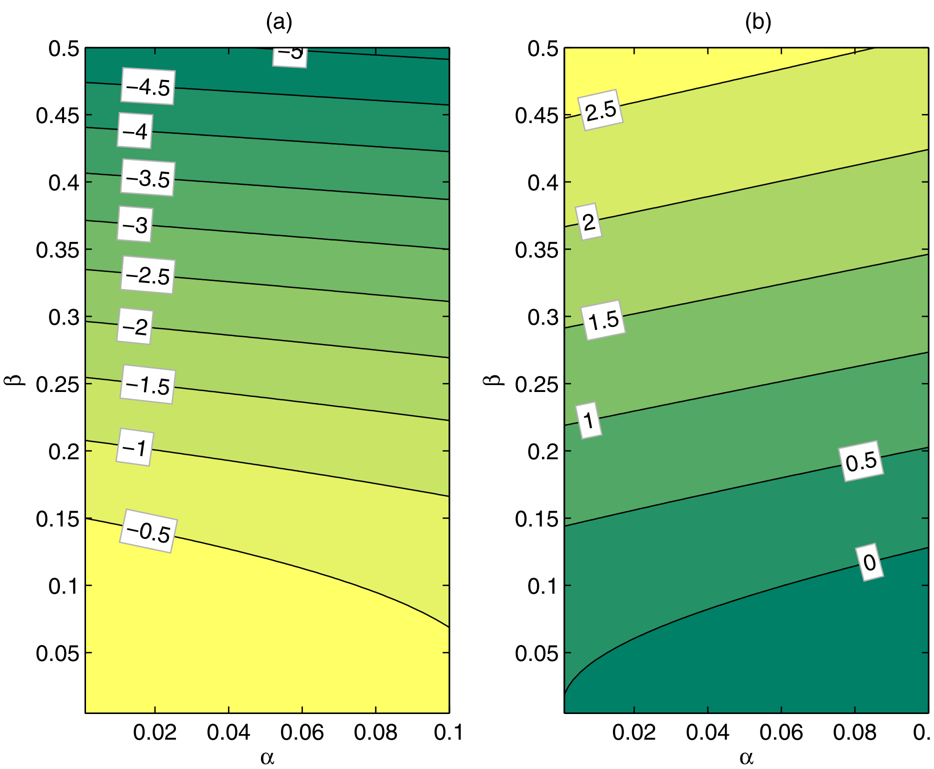


FIG. 3. Accuracy of the Padé approximant of length scale coefficient $e_0$ within the ranges of $0<\alpha\leq 0.1$ and $0<\beta\leq 0.5$: (a) $(e_0^{WF}/e_0^{W}-1)\%$ and (b) $(e_0^{RF}/e_0^{R}-1)\%$.

TABLE II. Calibration of $e_0$ for free vibration of CNTs with vibration mode $k=1$, $\alpha=0$ and Poisson ratio $\mu=0.19$.

| $L/D$ | $\beta$ | $e_0$ (the Wang model) | $e_0$ (the Reddy model) |
|---|---|---|---|
| 20 | 0.00308 | 0.409 | 0.407 |
| 12 | 0.00857 | 0.411 | 0.404 |

similar range of $e_0$ when compared with the values presented in Table I. For example, 0.31 and 0.39 from wave propagation in aluminum and KCI (Eringen and Edelen, 1972 and Eringen 1983), 0.288 from wave propagation in CNTs (Wang and Hu, 2005), and 0.2–0.6 from wave propagation in CNTs (Hu *et al.*, 2008). It demonstrates that the microstructured beam model may be used as an alternative benchmark to calibrate the length scale coefficient in the nonlocal theory.

## IV. CONCLUSIONS

The findings of this study are summarized as follows,

- For the vibration problem of discrete microstructured beam model with simply supported ends, exact solutions may be obtained as shown in Eq. (46). It is found that the shear stiffness (represented by shear springs) has a significant effect on the vibration frequency;
- The discrete microstructured beam model is continualized to its equivalent continuous system by Padé approximants. The analytical solutions of the equivalent continuous system are obtained as shown in Eq. (61) and are found to have the first order accuracy when compared with the exact solutions. As a general solving technique, Padé approximants have the potential in solving the microstructured model with other boundary conditions in addition to simply supported ends;
- An alternative way to calibrate the length scale coefficient of nonlocal beams is established. Analytical expressions of $e_0$ as shown in Eqs. (70)–(77) are obtained by comparing the frequency solutions from microstructured beam model and nonlocal Timoshenko beams. It is shown that the length scale coefficient $e_0$ is a function of the structural geometry, shear stiffness and vibration mode number. The established calibration approach could be used for the calibration of $e_0$ in plate like and shell like structures; and
- The length scale coefficient $e_0$ for the vibration of CNTs ranges from $1/\sqrt{6}$ to $1/\sqrt{12}$ depending on the geometry and vibration modes. For the free vibration of long CNTs, say, the aspect ratio $L/d \geq 20$, the Wang model and the Reddy model produce a similar length scale coefficient $e_0$ for the low frequency vibration. With shorter beams, the Reddy model provides a smaller $e_0$ when compared with the Wang model.

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